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SOME CRITICAL REMARKS ON ANALYTICAL REALISM

I

IN 1903 appeared Mr. Bertrand Russell's "Principles of Mathematics," a book which has attracted widespread interest. It had the merit of discussing in a fairly accurate and sometimes witty manner the fundamental mathematical disciplines, such as geometry, mechanics, arithmetic, and transfinite assemblages, and of attempting to relate these subjects to a system of philosophy, namely, the "pluralism" of Mr. G. E. Moore, "which regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities, with relations which are ultimate and not reducible to adjectives of their terms or of the whole which these compose."[1] Russell's treatment has been called *neo-realism.*[2] The mathematical advantage of this philosophical position, Russel maintains, is that, unlike most current philosophies, it allows mathematics to be true[3] in a sense which he has frequently sought to explain in various articles and which need not be dwelt upon further here.[4]

As to his method Russell says: "Our method will be one of analysis, and our problem may be called philosophical in the sense that we seek to pass from the complex to the simple, from the demonstrable to its indemonstrable premisses." Also Russell does not distinguish[5] between inference and deduction; induction appears[6] to him as "either disguised deduction or a mere method of making plausible guesses." It is natural enough that he should find the relation of

1 "Principles of Mathematics," page viii.

2 H. Dufumier, *Revue de Metaphysique et de Morale,* Vol. 17 (1909), page 620.

3 *Loc. cit.,* page viii.

4 *Cf.,* for instance, "The Problems of Philosophy," London; also Russell's "Philosophical Essays," New York, 1910.

5 "The Principles of Mathematics," § 1.

6 *Cf.* also *Monist,* Vol. 23 (1913), pages 489–490.

whole and part (which had been previously discussed by De Morgan) fundamental for his position; indeed, he says that for the comprehension of analysis it is necessary to investigate this notion.[7]

As might be expected from Russell's realism, his relational position is the so-called external one, which is opposed to the internal theories favored by idealists. An external relation is described as one implying no complexity in either of the related terms.[8] According to him[9] there exist external relations because asymmetrical relations are involved in Number, Quantity, Order, Space, Time and Motion, and it is impossible for him to explain asymmetrical relations on either of the usual theories of relation, *i. e.*, the monistic and the monadistic.[10] Concerning the external treatment of relations, it is important to recognize a contention made by Russell which, I think, has been supported[11] by Couturat: Russell implies that his discussion may serve as an engine of discovery in actual mathematics.[12]

Russell was able to carry out his discussion, as a whole, largely because he could avail himself of Peano's symbolic formulary of mathematics, whose principles he freely incorporated into his book and from whose mathematical content he drew much of his inspiration.[13]

II

In recent years, a group of young philosophers, "six realists" as they called themselves, has been engaged in controversy, notably with Dewey.[14] Presumably as an outcome of their activities they published a book last year, "The New Realism," consisting of six essays. One of these, "A Defense of Analysis," by Professor E. G. Spaulding, of Princeton, deals to a considerable extent with mathematics and will receive critical consideration in the present paper. Spaulding's purpose is to defend the general realistic interpretation of whole and part, to classify wholes into certain types, and to show that the analysis of each kind of whole does not lead to falsification.[15] The kinds of wholes Spaulding discusses are four:

I. Collections in numerical conjunction.

II. Classes formed or composed of parts which are not classes, but

[7] *Loc. cit.*, page 11, note; *cf.* A. T. Shearman, "The Development of Symbolic Logic," pages 203–205.

[8] *Loc. cit.*, chapter XVI.

[9] *Loc. cit.*, page 224, paragraph 1.

[10] *Loc. cit.*, § 216.

[11] *Cf. Monist*, Vol. 22 (1912), page 524.

[12] *Loc. cit.*, page 24, § 27.

[13] See Russell's own statements, *loc. cit.*, page 26, § 31, and elsewhere.

[14] This JOURNAL, Vol. VIII. (1911).

[15] "The New Realism," pages 155, 157, 168.

which may be either organic wholes or individuals or simples or collections.

III. Classes formed or composed of subordinate classes.

IV. Unities or organic wholes.

Each of these "wholes" or rather specific instances of the latter, Spaulding examines in turn, in order to establish the thesis that analysis is the discovery of the parts of a whole and the organizing relations which these parts sustain to each other.[16]

III

1. It is not my intention to consider Spaulding's essay point by point in detail and in the order of his article. I shall relate myself to crucial philosophic statements and then consider such mathematical errors as are typical.

A comparison of Spaulding's article with Russell's "Principles of Mathematics" shows at once that he has tried to carry out the Russell programme. Indeed, he has almost literally followed Russell in many instances, especially as to mathematics, and, when he deviates from Russell, frequently falls into errors of a very obvious nature.

One of the first points that Spaulding[17] makes is that "all the attacks on analysis are made by methods which themselves involve analysis or are analytical." In testing the effectiveness of any philosophical system, we have the right to investigate its utility (1) with reference to known contents, (2) with reference to unsolved problems. As to the second consideration, we are asked to take the word of the realists that their analysis is a means of discovery, but for tangible evidence they, in effect, refer us to analyses of known contents. Not being creative mathematicians themselves, it would be well for the realists to note what means of discovery eminent mathematicians employ to solve problems in research; but then they would find that the logical position of these investigators is very different from their analytical realism.[18] When in the presence of an unsolved problem even Russell abandons his method of analysis and becomes—as an examination will show—typically inductive; thus illustrating Dewey's remark[19] that a universal seems necessarily as-

[16] *Loc. cit.*, pages 158, 161, 168. It is hardly necessary to recall Aristotle's well-known comparison of a "whole" with an "organism."

[17] *Loc. cit.*, page 160.

[18] *Cf.* H. Poincaré *L'Enseignement Mathématique*, Vol. 10 (1908), pages 357–371. This position of Poincaré has been criticized by E. Borel, *Revue du Mois*, Vol. 7 (1909), page 98. Further, G. Cantor has stated his own general position to be Aristotelian realism. See *Zeitschrift für Philosophie*, Vol. 91 (1887), page 86.

[19] *Decennial Pub. of U. of C.*, Ser. 1, Vol. 3, page 122.

sociated with the existence of a problem. But even in the analysis of known contents the realists are unable to free themselves from inductive methods. Peirce has said that the syllogism involves an element of observation,[20] and a similar remark may be made of all deduction and of all analysis. It is also easy to recognize observational elements of Spaulding's discussion.[21] Spaulding compares the points on a line, the instants of time, and the series of real numbers and finds common properties. Similarly, the so-called "platform" which the six realists have printed as an appendix to their book aims to be a doctrine underlying the six essays which, as they frankly admit, are not in complete agreement. This must be the reply, then, to Spaulding's general criticism of the attacks on analysis.

2. Spaulding's attempted general refutation of the attacks on analysis is incidental to his consideration of specific arguments that analysis is identical with falsification. The instances of attacks upon analysis which Spaulding controverts, hardly do justice to the possibilities. One of these so-called model attacks[22] on analysis runs as follows: "The analysis of space leads to terms which are not spatial; it leads from the extended, the dimensional, to the unextended, the undimensional." Another attack[23] is stated thus: "Space . . . is given empirically by intuition (or some such mode of direct approach) as a unitary continuous whole. But analysis leads to terms or to parts of space which are discrete from one another." Spaulding does not seem to realize[24] that to regard mathematical space as the result of an analysis of a "whole" is itself a false attitude; such a position refers to a preliminary, perceptual space rather than a complete space as a concept. My contention is that the analysis of a "whole" could never induce the mathematical space Spaulding implies in his essay, but only a limited portion of this space. The problem of the genesis of one of those mathematical contents to which the generic name *geometry* is applied is, of course, classic.[25] In the heuristic development of space, as I conceive it, we have first presented to us a perceptual spatial content; in this perceptual "whole," "points" are perhaps spatial magnitudes, "lines" or "rods" are of limited length and have thickness, "planes" or "plates" are of limited area and have thickness, etc.[26] Comparison of such perceptual

[20] *American Journal of Mathematics,* Vol. 7 (1884–1885), page 182; *cf.* G. H. Mead, *Phil. Rev.,* Vol. 9, pages 5–9.

[21] For instance, on page 184 of his essay.

[22] Spaulding, *loc. cit.,* page 186.

[23] Spaulding, *loc. cit.,* page 186.

[24] *Loc. cit.,* pages 169, 187.

[25] *Cf.* Hölder, "Auschauung und Denken in der Geometrie," page 2.

[26] *Cf.* Veronese, "Grundlagen der Geometrie," pages 52–56, 225, 226, etc.

spaces leads to the organization of what may be called a fragmentary conceptual space. By a generalizing, constructive process in which the principle of complete induction plays an important part, this fragmentary space is completed. The completion may be effected in a variety of ways and, indeed, we have at least three geometries, *viz.*, the Euclidean, the Lobatcheffskian, and the Riemannian. Between such completed spaces and the naïve, perceptual contents which approximate the former, Spaulding has not properly discriminated; nor does he seem to understand that the term "space" in mathematics is rather superficial. In fact, "space" is a name applied to many collections of mathematical elements, but its use is a mere matter of convenience, not essential to mathematics.[27] As in the case of space, so in regard to continuity, time, and motion; Spaulding has not properly recognized the conceptual constructive systems as distinguished[28] from the crude percepts which led to them (in part), and these misunderstandings are quite sufficient to throw out any argument based on them. The thesis that Spaulding and the other realists should refute to show that analysis does not lead to falsification may briefly be stated thus:

Let it be granted that there exist infinite[29] wholes; then it is not possible to analyze such wholes without leading to contradictions.

Here is an opportunity for the realists to display their analytical skill. It seems that Russell's analysis resulted in an antinomy,[30] which is strangely at variance with his statement[31] that analysis gives us the truth and nothing but the truth.

3. In a future article I hope to discuss the nature of the whole-part relation, its hypothetical and intuitive significances, etc. I wish here merely to refer to that intuitive aspect of a whole which refers to the parts into which the whole has been analyzed. Granted that a whole has been successfully analyzed into parts, these parts represent a specific choice and this choice is intuitive. There must, then, have been a guiding principle associated with the analysis. Let me give an example. In the case of a descriptive three-space there are at least three analyses possible, each leading to an asymmetrical relation, namely, one-dimensional, two-dimensional, and three-dimensional. For each analysis there exists a specific definition of be-

[27] *Cf.* E. J. Wilczynski, *Bull. Am. Math. Soc.*, Vol. 19 (1913), pages 333–334.

[28] *Cf.* J. Royce, "The World and the Individual," Vol. I., pages 526–588.

[29] *Cf.* Spaulding, *loc. cit.*, pages 157, 201; Russell, *loc. cit.*, Chap. XVII.

[30] *Loc. cit.*, §§ 70, 78, 100, 344. This antinomy, by the way, like many others, seems to have as underlying problem the interdependence of object and act, clearly recognized by Plato, "Parmenides," 135, etc.; "Phædo," 73.

[31] *Loc. cit.*, page 141.

tweenness.[32] Russell and Spaulding have taken account of only one analysis and only one definition of betweenness. Now what is to guide us in adopting one analysis rather than another? Russell has touched[33] upon this question:

"It is important to observe that the definition of a space, as of most other entities of a certain complexity, is arbitrary within certain limits. . . . For example, in place of defining the line by a relation between points, it is possible to define the line as a class having a certain relation to a couple of points. In such cases we can only be guided by motives of simplicity."

It would be interesting to know what Russell[34] means by "motives of simplicity," and it is hard to see what test of simplicity of parts there can be other than "satisfactory functioning." This practical test, indeed, enables us to make a choice of the three analyses mentioned above. A three-dimensional analysis of three-space finds its justification in its relevance to the foundations of vector analysis and the application of the latter to mechanics and physics.[35] The geometric example just cited shows that analysis requires intuitive control, and this control must prevent irrelevant analyses or consideration of irrelevant contents. This deficiency suggests that Russell tends towards *scholasticism.* What Green[36] says of the Aristotelian logic is not without application to Russell and Spaulding:

"Thus the Aristotelian or syllogistic logic earns the reproach of consisting in a series of verbal propositions. It represents neither a method of arriving at knowledge nor the system of ideas which constitute the known world . . . but is merely of use in analyzing what is involved in conceded general propositions. . . . Hence its use by the Schoolmen. They did not want a method of arriving at truth nor a theory of what knowledge consists in. . . . As a rule for securing consistency in the interpretation and application of general terms, syllogistic logic has its value."

While, of course, Russell's position, like that of Boole and De Morgan,[37] occupies broader ground than the syllogistic logic, yet I think that the criticism just quoted suggests a fundamental defect of Russell and Spaulding. This view (as to Russell) finds support in

32 *Amer. Jour. of Math.,* 1909, page 365.

33 *Loc. cit.,* page 432.

34 *Cf.* "Principles of Mathematics," page 251, last lines; page 379, paragraph 4.

35 *Cf. Amer. Jour. Math.,* 1913, pages 37–56.

36 "Philosophical Essays," Vol. II., page 160.

37 Both Boole and De Morgan recognized the inadequacy of the syllogistic logic. *Cf.* "Laws of Thought," page 10, and *Camb. Phil. Trans.,* 1864, page 335.

an able survey[38] of Russell's "Principles of Mathematics" by Haussdorff. The latter says:

"A scholastic acuteness which perceives imaginary problems and neglects real difficulties, celebrates in Russell's book orgies of subtlety." Again, he says:

"In Russell's book are two conflicting tendencies, *viz.*, the formalistic, nominalistic, and one opposite to this for which it is difficult to find a name; an *a priori* tendency, realistic in the medieval sense, which would force us to discriminate, in a definite manner, between what is fundamental and what is derived and leads us to hair-splitting decisions in matters which are purely definitional."

To review briefly; the realistic position of Spaulding and Russell is insufficient to account for "wholes," in particular, those of a mathematical nature; it is inadequate, too, in the control of content if we admit that a whole has been successfully analyzed. The test of ultimacy of an analysis into parts must be found in the satisfactory functioning of these parts. For evidence of irrelevant analyses we have only to turn to Russell's book which seems to indicate ignorance on the part of the author of the practical needs of mathematics and logic.

4. The ultimacy of analysis which formed the subject of the preceding section suggests examination of the Russell-Spaulding treatment of asymmetrical relations and their general theory of relation. Spaulding,[39] citing Russell,[40] says that asymmetrical relations are unintelligible on any other theory than that of external relations; and to justify this statement Russell examines[41] the monistic and monadistic theories of relation and concludes, at least to his own satisfaction, that they are inadequate. Russell has given asymmetrical relations great prominence in his book. In every argument, if he has an opportunity, he leads his readers to an asymmetrical relation. The theory of magnitude, when based on transitive, symmetrical relations seems to Russell paradoxical and complicated; asymmetrical relations provide a simple and consistent theory; geometric order is generated by an asymmetrical relation and similarly in regard to time and motion.[42] Now the question may fairly be asked: Are asymmetrical relations indispensable from a practical standpoint? Of asymmetrical relations in general Royce[43] says:

"The contrast between symmetrical and unsymmetrical relations

[38] *Vierteljahrschrift für wiss. Phil. u. Soz.*, Vol. 29 (1905), pages 119–124.

[39] *Cf.* Spaulding, *loc. cit.*, page 176, note.

[40] Russell, *loc. cit.*, § 216.

[41] Russell, *loc. cit.*, §§ 212–215.

[42] *Cf.* Russell, *loc. cit.*, §§ 154–157; 206–207; 441, 446.

[43] *Transactions of the American Mathematical Society*, Vol. 6 (1905), pages 358–359.

seems, to the ordinary view, absolute. Mr. Russell, in his late volume, so treats it. . . . In symbolic logic, however, a symmetrical copula, namely, that of 'inconsistency' or of 'opposition' can be made to accomplish all the work of the ordinary unsymmetrical copula $-\!\!<$. In other words, if I have otherwise defined the meaning of 'not,' the statement 'x is inconsistent with not-y' means the same as 'x implies y.' The copula in the former case is symmetrical, in the latter unsymmetrical."

But also mathematically there is no valid reason why we should regard an asymmetrical relation more ultimate than a symmetrical one. A line may be generated by a transitive asymmetrical relation[44] between points, *i. e.*, a relation of the type

$$aRb \text{ implies not } bRa,$$
$$aRb \text{ and } bRc \text{ imply } aRc,$$

or a transitive symmetrical relation[45] between dyads, *i. e.*, a relation of the type

$$abKcd \text{ implies } cdKab,$$
$$abKcd \text{ and } cdKef \text{ imply } abKef.$$

Russell curiously infers from the definition of asymmetrical relations on the basis of the symmetrical that the latter are not essential.[46] Why? By applying an analogous argument to asymmetrical relations we might easily prove that these are non-essential. For example we can define[47]

$$abKcd \text{ means } (aRb \text{ and } cRd) \text{ or } (bRa \text{ and } dRc).$$

In the second case we have transitiveness and symmetry on the basis of asymmetrical relations. In geometry we should say that the two methods of generating space are equivalent. If we consider n-dimensional space generated linearly ($n > 1$) an infinite class of transitive asymmetrical relations is required,[48] while a single transitive symmetrical relation suffices. Thus there exists in this case a practical reason for preferring transitive symmetrical relations. Russell's reduction of the latter to asymmetrical relations, by the way, is effected through a "principle of abstraction," closely allied to Peano's "definition by abstraction" which Vailati has characterized as *pragmatic*.[49] From a practical mathematical standpoint I am unable,

[44] *Cf. American Journal of Mathematics*, Vol. 31, page 378.

[45] *Cf. Amer. Jour. of Math., loc. cit.*, page 394.

[46] *Cf.* Russell, *loc. cit.*, page 235.

[47] *Cf.* Russell's view of "and" as a relation, *loc. cit.*, page 71.

[48] *Cf.* Russell, *loc. cit.*, page 395.

[49] *Cf.* G. Vailati, "Pragmatism and Mathematical Logic," *Monist*, Vol. 16, page 487. Russell uses the principle of abstraction throughout his book; see espe-

then, to verify that absolute position asymmetrical relations enjoy in Russell's book.

5. Before considering the possibility of constructing an internal theory of relations, including the asymmetrical, it will be useful to exhibit some of the inconsistencies and vagaries in the Russell-Spaulding external theory. We notice, for example, an arbitrariness and uncertainty on the part of Russell and Spaulding concerning particular relations. Russell says: "It seems best to regard *and*[50] as expressing a definite unique kind of combination, not a relation." Spaulding,[51] on the other hand, assumes that *and* does express a relation. Again Russell explicitly assumes[52] that membership of a term in a class is a relation, and this assumption leads[53] him to affirm that some relations which hold between a term and itself are not necessarily symmetrical, a statement which seems formally undesirable. Concerning identity Russell[54] says frankly:

"The question whether identity is or is not a relation and even whether there is such a concept at all[55] is not easy to answer. For it may be said identity can not be a relation since where it is truly asserted we have only one term, whereas two terms are required for a relation. . . . Identity must be admitted and the difficulty as to the two terms of a relation must be met by a sheer denial that two different terms are necessary." The conclusion that must here be drawn is that identity, as a relation, has a very dubious existence. And if identity, as a relation, is in question, the same must be said of difference, because the interdependence of identity and difference is, I think, fairly well recognized.[56] Lastly, I observe that Russell has arbitrarily assumed[57] that a sensed couple involves a relation:

"It may be doubted whether there is any such entity as the sensed couple, and yet such phrases as '*R* is a relation holding from *a* to *b*' seem to show that its rejection would lead to paradoxes."

"It would seem, viewing the matter philosophically, that sense

cially *loc. cit.*, page 519; compare also page 51. See also G. Vailati, *Revue du Mois*, Vol. 3, 1907, pages 162–185.

50 The term "and" has a far more pregnant meaning in symbolic logic than Russell recognizes (*loc. cit.*, §§ 71, 98). Consider, for instance, the definition, given above, of the relation of *abKcd* in terms of the relation *aRb*.

51 *Loc. cit.*, page 162.

52 *Cf.* Russell, *loc. cit.*, §§ 21, 26, 30, 53, 68, 69, 76–78, 125, (*cf.* 491); see also pages 25, 167.

53 *Cf.* Russell, *loc. cit.*, §§ 30, 57, 76, 79, 94, 95.

54 *Loc. cit.*, pages 63–64.

55 On page 96, *loc. cit.*, Russell says: "Self-identity is plainly a relation," but on page 163 expresses doubt about identity being a relation.

56 *Cf.* for instance, Bradley, "Appearance and Reality," 2d edition, pages 585, 617, etc.

57 *Loc. cit.*, pages 87–88; 99 (*cf.* page 25); 512, note; 107, note.

can only be derived from some relational proposition." Russell thus seems by no means certain that the sensed couple involves a relation.

If one assumes that identity and diversity are not relations,[58] that possession of a trait does not express relation,[59] that reference[60] to a term is non-relational, and that no relation is involved in a sensed couple or rather *functional ordered*[61] *dyad* $(x\ y)$, then it seems possible to explain asymmetrical relations on an internal basis. I will mention briefly how this might be done. As a standard form of a binary functional relation, I assume xRy, that is, "x possesses R with reference to y;" a relation between x and y arises, then, if the term x possesses a *mark* or *trait* with reference to the term y. Now I assume that xRy is always equivalent[62] to $(xy)R_1(xy)$ where (xy) is a functional ordered dyad, and that R_1 is symmetrical, *i. e.*, (xy) and its repetition may be interchanged. Therefore the preceding interpretation suggests that a binary relation may be generated by comparing an ordered dyad (xy) with its repetition; one has xRy if, and only if, (xy) possesses R_1 with reference to itself, or (xy) and its repetition possess a common mark or[63] the dyad (xy) and its repetition are "relatively equal" with reference to a mark. It should be observed that the equivalence of xRy and $(xy)R_1(xy)$ involves subtle distinctions in Russell's external theory of relation; it has as underlying problems the analysis of a reflexive relation[64] and the relation of the class of all propositions of the form xRy to the associated propositional function of two variables, $\phi(x, y)$.[65] This preceding internal theory seems consistent, but contradicts[66] several

58 *Cf.* Bradley, *loc. cit.*, page 582.

59 Russell, *loc. cit.*, §§ 53, 79, also §§ 425, 426.

60 Russell, *loc. cit.*, § 214. In the above I conceive of "trait" not merely as something that may be possessed by a term, but also as something that is *relevant* to some term (*cf.* Russell, *loc. cit.*, §§ 81, 82). Reference, or rather relevance, is a preliminary that may lead to relation. On "relevance" see Schiller, *Mind,* 1912.

61 See *Amer. Jour. of Math.*, Vol. 31, pages 370, 375.

62 As tending to illustrate this equivalence consider "$x = y$" and "$x - y = x - y$, $x - y = 0$."

63 *Cf.* Veronese, "Grundzüge der Geometrie," pages 2–5.

64 *Cf.* Russell, *loc. cit.*, page 86, paragraph 2.

65 It seems possible to approach the above equivalence on the Russell basis by saying (*cf.* Russell, *loc. cit.*, page 85, paragraph 2 and § 74) $(xRy)\epsilon\phi(x, y)$, or if one is thinking of a functional ordered dyad underlying xRy one has $(x, y)\epsilon\phi(x, y)$, as equivalent to the former symbolic statement. Now $(x, y)\epsilon\phi(x, y)$ expresses a relation of a class to a class of which it is the only member, *viz.*, a class of couples (x, y) to the associated propositional function $\phi(x, y)$. One might, therefore, conveniently express $(xy)\epsilon\phi(x, y)$ in terms of a single (relational) symbol and get $(xy)R_1(xy)$. Compare also A. T. Shearman, *Mind,* 1907, page 260.

66 I assume that subject-predicate propositions are reducible to the standard,

of Russell's controversial assumptions concerning relation and class.[67]

It seems plausible, therefore, to modify the monistic and monadistic theories of relation so as to yield, formally at least, an unobjectionable internal[68] theory. Against Russell's external theory, it might be urged that the external theory assumes[69] the definition of the general effectiveness of a relation:

"The relation affirmed between *A* and *B* in the proposition '*A* differs from *B*' is the general relation of difference and is precisely and numerically the same as the relation affirmed between *C* and *D* in '*C* differs from *D*.' And this doctrine must be held to be true of all other relations; relations do not have instances, but are strictly the same in all propositions in which they occur."

In my opinion a more correct statement would be that general concepts of relation are limit concepts to which classes of specific instances of relation sometimes tend. Aside from this, however, it may be questioned whether Russell has succeeded in entirely avoiding relations as specific instances. What I suspect to be a disguised internal relation is Russell's "measurable relation between two vectors" which he describes[70] as follows: "To say that the relation is measurable in terms of real numbers means . . . that all such relations have a (1, 1) relation to some or all real numbers." From the standpoint of the correspondence with real numbers it seems altogether likely that we are concerned here with specific relational instances in which the terms related are peculiarly involved. A precisely analogous relational problem occurs elsewhere in the abstract mathematical science of Grassmann, the *Ausdehnungslehre* which has geometry and mechanics for particular applications. On the introduction of the number system into his discipline Grassmann states[71] explicitly: "The numerical magnitude as developed in our science does not appear as discrete number, *i. e.*, not as a set of units, but . . . as a quotient of continuous magnitudes and therefore does not at all presuppose the discrete conception." This conception of number, regarded in a certain way as a foundation for a general theory, may have its limitations; nevertheless, its use is, I think,

equivalent forms, "*a* possesses *M*," "*a* belongs to *C*," "*O* affects *a*," where *M, C, O* may lead to a relation, class, operation, respectively. See Russell, *loc. cit.*, §§ 57, 79.

67 See, for instance, "Principles of Mathematics," page 167, paragraph 1.

68 I do not wish to imply here that I uphold a purely internal theory of relation. On the contrary neither an internal theory nor an external theory, in itself, appears to me adequate.

69 Russell, *loc. cit.*, § 55, page 51.

70 *Loc. cit.*, page 433.

71 *Cf. Gesammelte Werke*, Vol. I., page 138.

amply justified in Grassmann's Extensive Algebra and constitutes a serious difficulty in the purely external theory of relation.

6. Reverting for a moment to the falsification of analysis, I may be permitted to indicate how conflict, which is the source of embarrassment to analytical realists, is employed to advantage in the pragmatic position. To have recognized the fundamental part of conflict in the process of knowledge is, I believe, one of the great merits of pragmatism.[72] This philosophy, at least in regard to conflict, seems more nearly in accord with the facts of mathematics as an incomplete science than analytical realism. Let me review briefly a few important instances of conflict in the history of mathematics and the developments to which they have given rise. It is proper here to quote Hilbert:[73]

"In modern mathematics the question of the impossibility of solution of certain problems plays an important rôle and the attempts made to answer such questions have often been the occasion of discovering new and fruitful fields for research. We recall . . . the demonstration by Abel of the impossibility of solving an equation of the fifth degree by means of radicals, as also the discovery of the impossibility of demonstrating the axiom of parallels, and finally the theorems of Hermite and Lindemann concerning the impossibility of constructing by algebraic means the numbers e and π."

Again, Hamilton endeavored to construct an algebra of three units, $a + ib + jc$, which should obey the same laws of operations as the ordinary complex number, $a + ib$; and out of the conflicts that arose between these algebras, as he describes in detail in the preface to his "Lectures on Quaternions," he was led to construct a new complex number of four units, the quaternion. If I may give another example, the problem of the continuity of the straight line is that presented by the conflict of an intuitive straight line, say L, with the class R of rational numbers; namely, on the straight line L there is an arbitrary number of points which corresponds to no rational number, while to every rational number there corresponds a point. Thus as Dedekind says in his celebrated memoir,[74] a comparison of the intuitive straight line L with the rational numbers R shows that the

[72] *Cf.* G. H. Mead, *Philosophical Review*, Vol. 9; A. W. Moore, "Pragmatism and its Critics," page 125. See also Stosch, *Vierteljahrschrift für wiss. Phil.*, Vol. 29, page 97, note 3.

[73] "Foundations of Geometry," page 131. Compare O. Perron, "Ueber Wahrheit und Irrthum in der Mathematik," *Jahresber. d. Deutsch. Math. Ver.*, Vol. 20 (1911), page 196; H. Liebmann, "Nothwendigkeit und Freiheit in der Mathematik," same journal, Vol. 14 (1905), page 230.

[74] "Stetigkeit und irrationale Zahlen," pages 7–11.

latter presents gaps, but the former does not. Dedekind's solution of the conflict was his formulation[75] of the principle:

"If all the points of a line are separated into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this separation." More generally, I might refer to the conflict between analysis and geometry in the development of mathematics and that great movement initiated by Lagrange known as the arithmetization[76] of mathematics in which rival theories due to Cauchy and Weierstrass and Méray are prominent. But the examples given will, I think, sufficiently indicate an important aspect of mathematics as heuristic.[77]

IV

I come now to criticisms of a more properly mathematical nature. One of Professor Spaulding's colleagues, Professor W. B. Pitkin, in the "New Realism" (p. 378), speaks of the objections which mathematics has brought against realism in the past, and implies that these have been cleared away in previous essays in the volume, presumably Spaulding's essay. This statement, alas! can not be verified. Spaulding has committed many mathematical errors and it is proper to state that these seem due to an unfamiliarity with mathematical conceptions, rather than to the peculiar philosophic position he upholds; his mathematical remarks obscure rather than elucidate his fundamental theses. On this account I enter upon a mathematical criticism of Spaulding's essay rather unwillingly, and a few indications of the most noticeable mistakes must suffice.

Readers of Spaulding's essay will quite agree with the author when he says,[78] "It is important . . . to present clearly and with precision that which analysis shows the continuum to be." But this is what Spaulding does not do. (1) Consider his statement (p. 178), "That there are irrationals is discovered in the realization that there

[75] Obviously, Dedekind's statement lacks rigor; *cf.* Russell, *loc. cit.*, § 266.

[76] *Cf.* G. Bohlmann, *Jahresbericht der Deutschen Mathematiker Vereinigung*, 1901, page 95.

[77] In the preceding section it might have been desirable to dwell on the nature of conflict in general. Research mathematicians will probably have no difficulty in recognizing in their own experiences what is meant by the *conflict* of mathematical terms. The subtle character which such conflict often possesses may be illustrated by an example. The statements, "$1+2=3$" and "$9+16=25$" are not in conflict in reference to addition of integers, but they are in conflict in that $9+16=25$ may be expressed $3^2+4^2=5^2$, while $1+2=3$ does not admit an analogous expression in terms of squares. More broadly, there is a conflict between "intuitive" and "formal" mathematics.

[78] *Loc. cit.*, page 78.

is some value for x whereby, for example $x^2=2$. The position[79] that underlies this statement is incorrect. What probably suggested this remark to Spaulding is the possibility of making the (intuitive) construction which he has described (p. 184, paragraph 3).[80] (2) The definition of limit of a sequence Spaulding misquotes[81] from Pierpont's book.[82] I remark that Pierpont's theory of real numbers has been discussed in a review by G. A. Bliss.[83] (3) The error is committed by Spaulding (p. 179) of juxtaposing the *derivative of an assemblage* and the *derivative of a function;* further comment on this seems unnecessary.

Spaulding's analysis of space is not more satisfactory than his arithmetical analysis. (1) Spaulding seems to have been misled (p. 184, paragraph 1,) by Hilbert's use of the word "continuity" in connection with the Archimedean property of a line.[84] (2) The author's inability to comprehend mathematical continuity is clearly shown, as is evident elsewhere, by his remark (p. 185) that a "series" is continuous if it is perfect. The latter remark is contradicted by *nowhere dense perfect assemblages.*[85] (3) Spaulding makes a very feeble attempt (p. 188) to explain the relation of the extension of a line to its continuity. What we are concerned with here is the dependence of the Archimedean axiom on the axiom of Dedekind continuity in the foundations of geometry; this has been recently discussed by O. Hölder.[86] In view of such instances as the preceding, Spaulding is not justified in saying[87] that his analysis of space "states with clearness and precision what space is, what its continuity is, what terms and relations are involved."

I shall not consider further Spaulding's errors. Enough has been

[79] Russell has a remark quite as misleading as the one quoted above from Spaulding; see Jourdain, *Math. Gazette,* Vol. IV. (1908), page 204, note.

[80] Compare also Dedekind, *loc. cit.*

[81] *Loc. cit.,* page 178.

[82] "The Theory of Functions of Real Variables," Vol. I., page 25, § 42; page 61, § 97.

[83] *The Bulletin of the American Mathematical Society,* Vol. 13 (1906–1907), pages 121–122. See especially Bliss, *loc. cit.,* page 121, note; compare H. Weber, *Jahresber. d. Deutsch. Math. Ver.,* Vol. 15 (1906), page 173. Instructive references to the theory of real numbers are, A. Pringsheim, preceding journal, Vol. 6, page 73; O. Perron, same journal, Vol. 16 (1907), page 142, and Jourdain, *Math. Gazette,* Vol. IV. (1908), page 201.

[84] "Foundations of Geometry," page 24.

[85] See Schoenflies, *Jahresbericht der Deutschen Mathematiker-Vereinigung,* Vol. 8 (1900), pages 101–102; *cf.* Russell, *loc. cit.,* page 440, § 417; page 288, § 272.

[86] *Leipziger Berichte,* Vol. 63 (1911), pages 108–109. *Cf.* K. Th. Vahlen, *Jahresber. d. Deutsch. Math. Ver.,* Vol. 16 (1907), page 409.

[87] *Loc. cit.,* page 185.

said to establish that he not only is unable to analyze successfully those mathematical contents with which he has dealt, but he does not even possess familiarity with the mathematical conceptions he mentions. In this respect Spaulding differs from Russell, who, I think, has been reasonably accurate in dealing with mathematical conceptions, and who has been aptly characterized by James[88] as an "athletic ratiocinator." But so far as I have been able to ascertain, neither Spaulding nor Russell has had experience in mathematical research, and discovery, as Peirce[89] has pointed out, is a part of mathematics.

V

In conclusion, I should like to lay stress on the desirability of a more intimate relation between philosophers and mathematicians. "The mathematician's interests," says Royce,[90] "are not the philosopher's. But neither of the two has a monopoly of the abstractions and in the end each of them—and certainly the philosopher—can learn from the other. The metaphysic of the future will take fresh account of mathematical research." The numerous misinterpretations of mathematics occurring constantly in philosophical literature, probably not excepting the work of Bergson,[91] show that philosophers can not pronounce judgment on mathematical contents without acquainting themselves with mathematics in a way that probably requires actual mathematical experience. Conversely, mathematicians should endeavor to enter into the spirit of philosophical disciplines and recognize that the study of philosophy can be made indirectly the means of further mathematical development.[92] Mathematical masters have sometimes acknowledged explicitly this advantage of philosophic study. It is said of Kronecker,[93] for instance, that he thought more philosophically than mathematically and considered it profitable to go beyond his special mathematical field, to aim at general ideas, and then to return to his more restricted activity.

ARTHUR R. SCHWEITZER.

UNIVERSITY OF CHICAGO.

88 "The Meaning of Truth," page 276.

89 *Cf.* J. B. Shaw, *Bull. Am. Math. Soc.*, Vol. 18, page 381.

90 "The World and the Individual," Vol. I., page 527; compare Vol. II., page x.

91 *Cf.* E. Borel, *Rev. de Met. et de Morale,* Vol. 16, 1908, pages 244–245.

92 *Cf.* M. Winter, *Rev. de Met. et de Morale,* Vol. 16, 1908, page 920.

93 *Cf.* Netto, Mathematical Congress Papers, Chicago, 1893, page 243; see also page 246.